Probabilistic Approach for Best Estimate of Fuel Rod Fracture

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Abstract: Existing risk assessments for nuclear power plants have been refined in terms of plant response analyses and system analyses. However, the criteria for determining core damage rely on deterministic criteria that are conservatively set, leading to an imbalance in the assessments. To achieve a more reasonable and realistic core damage determination, this study proposes a probabilistic approach that integrates a fuel rod fracture probability estimation model with the best estimate plus uncertainty analysis of plant response for loss-of-coolant accident (LOCA) conditions, considering uncertainties. Our proposed approach provides a probabilistic fuel rod fracture determination method using the stress-strength model and Monte Carlo simulations. Both the stress and strength distributions are estimated using Bayesian statistical modeling. To evaluate the effectiveness of our proposed approach, we conduct numerical experiments and compare the results with the existing deterministic approach. The results of the numerical experiments show that the proposed approach provides a more detailed and realistic fracture probability of fuel rods than the existing deterministic approach, eliminating conservatism. Furthermore, we explore numerical integration to enhance accuracy of the fracture probability estimation for low frequency events, offering an alternative to Monte Carlo simulations which might not effectively handle these events. Our proposed approach enables a shift from the conservative evaluation that equates a fuel rod fracture directly with core damage to a more realistic evaluation.

Keywords: Fuel rod fracture, PRA, BEPU, Bayesian inference, LOCA

1. INTRODUCTION

In the safety assessment of nuclear power plants, there has been progress in rationalizing methods such as best-estimate plus uncertainty (BEPU) and probabilistic risk assessment (PRA). However, these methods predominantly target the evaluation of stress side parameters, such as fuel cladding temperature. In contrast, the criteria for determining core damage, which is strength side parameters, are set conservatively and deterministically based on conditions ensuring that fuel rods (fuel cladding tubes) do not fracture during loss-of-coolant accidents (LOCAs) [1]. However, core damage occurs not merely from the fracturing of a fuel rod, but rather when such fractured rods collapse and blocks the core cooling. Consequently, the traditional safety assessment is overly conservative in determining core damage and lacks balance with stress side evaluations.

In response to this issue, a previous study has proposed a fuel rod fracture probability estimation model that provides the best-estimate of fracture probability of fuel rods, including uncertainties, using the amount of oxidation of the cladding tube as an explanatory variable [2]. This model provides the best estimate of the strength side parameter. However, an integrated study applying the BEPU approach to both the stress side and the strength side has not been conducted yet.

To establish a more rational and realistic core damage determination, this study explores a probabilistic approach that integrates the fuel rod fracture probability estimation model with the plant response analysis using the BEPU approach for LOCAs of a light water reactor. Our proposed approach provides a probabilistic fuel rod fracture determination method using the stress-strength model and Monte Carlo simulations. To evaluate the effectiveness of our approach, we conduct numerical experiments and compare the results with the existing deterministic approach. Additionally, we explore numerical integration to enhance the accuracy of the fracture probability estimation for low frequency events, offering an alternative to Monte Carlo simulations, which might not effectively handle these events.

This paper is organized into five chapters. Chapter 2 explains the proposed fuel rod fracture determination method and the model's construction, citing previous studies. Chapter 3 shows the numerical experiments that compare the proposed method with the traditional deterministic method. Chapter 4 explores the use of numerical integration for the determination. Chapter 5 concludes with the achievements of this study, discusses prospects, and explores the potential application of this method to PRA, considering both its benefits and challenges.

2. PROBABILISTIC FRACTURE DETERMINATION METHOD

2.1. Fracture Determination Using the Stress-Strength Model

In this study, we propose a fracture determination method utilizing the stress-strength model and Monte Carlo simulations. This model employs probability distributions for both the stress side (the plant response analysis) and the strength side (the fuel rods fracture limit). In implementing this model, the probability distributions for both sides are estimated using the equivalent cladding reacted (ECR) as an explanatory variable during LOCA. The selection of ECR as an explanatory variable is justified by a previous study indicating that ECR is a predominant factor in the fracture of fuel cladding tubes during LOCAs [2].

We present herein the complete methodology of the probabilistic fracture determination approach proposed in our study. Initially, the distributions, which include uncertainties for the ECR on both the strength and stress sides, are estimated. Subsequently, values of ECR are sampled from each distribution, and the two are compared. If the ECR value derived from the stress side exceeds that from the strength side, a fracture is deemed to have occurred; otherwise, no fracture is determined. This sequence of trials is repeated sufficiently to calculate the fracture probability as the proportion of trials in which a fracture is determined to have occurred.

2.2. Estimation of Stress Distribution

To estimate the stress distribution, data on the ECR from specific accident scenarios are necessary. This study utilized the results from the BEPU analysis of a pressurized water reactor (PWR) large break LOCA (LBLOCA) scenario conducted by Zugazagoitia et al.[3] The BEPU analysis used the TRACE5 code, Patch4 [4], assuming a guillotine break at both ends of the reactor coolant system. A total of 1021 simulations were performed, estimating the peak cladding temperature (PCT) and the oxidation amount, referred to as the localized mass oxidation (LMO), which is defined identically to ECR. For the proposed methodology, the data on the oxidation amount calculated in the earlier study was utilized because our focus is on analyzing the relationship between the amount of oxidation (ECR) and fuel rod fracture.

Subsequently, using the prepared dataset, the stress distribution was estimated including uncertainty. A parametric estimation was performed assuming a log-normal distribution.

The log-normal distribution is chosen by its ability to only assume positive values, ensuring the distribution's tail remains within the positive range. Additionally, its long right tail facilitates the estimation of low-frequency, high-ECR values. These characteristics make the log-normal distribution particularly suitable for representing the probability density distribution of ECR data, which includes uncertainty. This suitability is the primary reason for choosing the log-normal distribution in this study.

For the parametric estimation of the distribution using the log-normal distribution, we use Bayesian inference with Markov chain Monte Carlo (MCMC) methods. The formula for this is expressed as follows:

$$X \sim LogNormal(\mu, \sigma) \tag{1}$$

Here, X represents the literature value of ECR (-), and μ and σ represent the mean and standard deviation of the log-normal distribution.

Using the posterior distribution of the parameters estimated from equation (1), the posterior predictive distribution of the stress side's ECR can be expressed as follows:

$$f(x \mid X) = \int \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\log x - \mu)^2}{2\sigma^2}\right) p_{post}\left(\mu, \sigma \mid X\right) d\mu d\sigma \tag{2}$$

where $f(x|X)$ represents the posterior predictive distribution of the stress side's ECR, x represents ECR, and $p_{post}(\mu,\sigma \mid X)$ represents the joint posterior distribution of the parameters.

The estimation involved running four chains with 2,000 iterations each, totaling 8,000 iterations of MCMC sampling. The first 1,000 iterations in each chain were discarded as warm-up, effectively generating a total of 4,000 MCMC samples. For the marginal prior distributions of the parameters, non-informative distributions were used, employing a normal distribution with mean 0 and variance of 10^4 [5].

The results are illustrated in Figure 1, where the blue histogram represents the ECR dataset from the simulation results of the previous study. The black line and shaded regions indicate the median, 50% interval, and 95% interval of the posterior predictive distribution, respectively.

2.3. Estimation of Strength Distribution

The strength distribution was estimated using a fracture probability estimation model developed in a previous study [2]. In that study, LOCA-simulated tests were conducted under conditions that eliminated conservatism. Using the resulting data on fracture and non-fracture of fuel rods with Zircaloy-4 cladding tubes, the relationship between ECR and fracture probability was modeled using Bayesian inference. Assuming that the binary fracture/non-fracture data follows a Bernoulli distribution, the log-probit model was used to construct the fracture probability estimation model. In this model, the ECR is calculated using the Baker-Just equation [6]. However, for the stress side, the ECR data from the simulation results of the previous study was derived using the Cathcart-Pawel equation [7]. To standardize the calculation methods of ECR for both the stress and strength sides, this study recalculates the ECR of the strength side using the Cathcart-Pawel equation to align with the stress distribution estimation.

The fracture probability estimation model is expressed as:

$$Y \sim Bernoulli(P(Y=1 \mid X)) \tag{3}$$

$$P(Y=1 \mid X, D) = \int \Phi\left[\alpha + \beta \log X\right] p_{post}(\alpha, \beta \mid D) d\alpha d\beta \tag{4}$$

where Y represents binary coded LOCA-simulated test data where 1 indicates fracture and 0 indicates no fracture. X is the ECR (-), $P(Y=1 \mid X, D)$ represents the posterior predictive distribution of the strength side's ECR, the link function Φ employs the cumulative distribution function of the standard normal distribution, α and β represent the scalars of unknown parameters to be estimated, and $p_{post}(\alpha, \beta \mid D)$ represents the joint posterior distribution of the parameters.

In this model, the parameters α and β were estimated using Bayesian inference with the MCMC method. For this estimation, 4 chains were run with 2,000 iterations each, totaling 8,000 MCMC samples. The first 1,000 iterations of each chain were discarded as warm-up, effectively generating 4,000 MCMC samples. The marginal prior distributions for the parameters were assumed to be non-informative, specifically a normal distribution with a mean of 0 and a variance of 10^4 [5].

The results are illustrated in Figure 2, where the red points represent the binary data concerning fracture and non-fracture of the test rods obtained from the LOCA-simulated test. The black line and shaded regions indicate the median, 50% interval, and 95% interval of the posterior predictive distribution of the fracture probability, respectively.

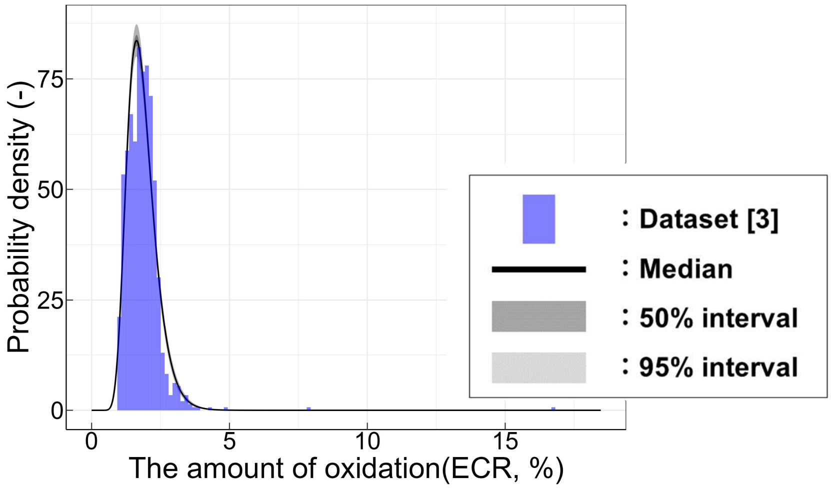

Figure 1. Probability Density Distribution of ECR Estimated using the Log-normal Distribution

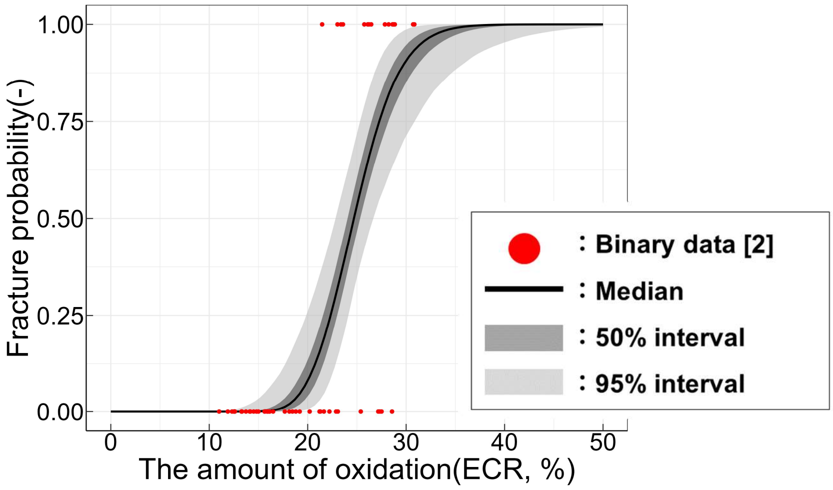

Figure 2. Fracture Probability Curve Estimated Using the Log-probit Model

3. TRADITIONAL VS. PROBABILISTIC METHODS

3.1. Analytical Conditions

In this section, we detail the traditional fracture determination method as a preliminary step before comparing them with the proposed method. Traditionally, the fracture determination is made by comparing the 95th percentile values, obtained through order statistics, with deterministic criteria. The principle behind order statistics involves arranging the safety evaluation parameters from most stringent to least stringent and using a defined number of analysis cases and the order to position the extracted analysis results outside a specified percentile at a designated confidence level. These relationships are represented by Wilks' formula [8], shown as follows:

$$\sum_{k=0}^{N-p} \frac{N!}{(N-k)!k!} \alpha^k (1-\alpha)^{N-k} \geq \beta \tag{5}$$

where both N and p are integers, α represents the percentile, β represents the confidence level, N represents the number of analysis cases, and p represents the order that specifies which parameter is selected based on its position in the sequence.

By substituting 0.95 for both the percentile α and the confidence level β, formula (5) simplifies to a relationship solely between N and p. Conventionally, to minimize the sample size, p is set to 1, which requires a sample size of 59, thus the largest value among the 59 samples is used as the 95/95 upper tolerance limit.

The following will detail the conditions under which both the traditional and proposed methods will be compared. As shown in Chapter 2, the proposed probabilistic fracture determination method can be divided into three steps: estimation of the stress side, estimation of the strength side, and the determination of fracture (calculation of fracture probability). Therefore, the comparison between the proposed and traditional methods will be segmented into these three perspectives.

Note that the BEPU results of the PWR LBLOCA scenario in the previous study [3] are unsuitable for comparing fracture determination methods because the ECR values are extremely low. Hence, in this section, a virtual dataset with the ECR values increased tenfold is used for comparison. From this dataset of 1021 entries, one entry where the ECR exceeded 100% was excluded, resulting in a virtual dataset of 1020 data points.

3.1.1. Estimation of Stress Side

In the proposed method, we estimate the stress distribution incorporating uncertainty from the dataset. Conversely, the traditional method uses order statistics to calculate the 95/95 upper tolerance limit, which serves as the representative value of the dataset for stress side estimation. To compare these two approaches, we calculated the 95th percentile values using each method and investigated which percentile of the original dataset these values correspond to.

To align with the commonly used first order in order statistics, we conducted the comparison using a sample size of 59. These 59 samples were randomly selected from the virtual dataset of 1020 entries. To assess the variability due to the randomness in sample selection, we repeated the trial to calculate the 95th percentile value 10^5 times, using the 59 sampled data points each trial. For each method, we calculated the average and standard deviation of the 95^{th} percentile values. We then evaluated which percentile in the original dataset corresponded to the average values obtained from each approach.

3.1.2. Estimation of Strength Side

In the proposed method, we estimate the fracture probability curve as a function of ECR using experimental data. This distribution is then used to sample multiple ECR values for the stress-strength model. In contrast, the traditional method employs a deterministic fracture limit where a fracture is assumed if ECR exceeds 15%. This 15% ECR threshold is based on the Japanese emergency core cooling system (ECCS) acceptance criterion [1].

In the ECCS acceptance criterion, the ECR is calculated using the Baker-Just equation [6]. However, the virtual ECR data utilized in this study is derived using the Cathcart-Pawel equation [7]. Consequently, for the purposes of this study, the 15% ECR threshold was converted to an ECR value calculated with the Cathcart-Pawel equation, adjusting the threshold to 11.6% ECR. To avoid confusion, the threshold will continue to be referred to as “15% ECR”.

To compare the probabilistic fracture limit (fracture probability curve) with the deterministic fracture limit, we investigated the fracture probability corresponding to the 15% ECR threshold on the fracture probability curve.

3.1.3. Fracture Determination

Fracture determination was conducted using two approaches: calculating fracture probability via Monte Carlo simulation and the stress-strength model, and referencing the deterministic criterion. The results of these methods were compared to assess their effectiveness. For the proposed method, the number of trials for the Monte Carlo simulation was set at 10^8.

3.2. Results and Discussion

3.2.1. Estimation of the Stress Side

Table 1 shows the comparison results of 95th percentile of the stress distribution estimated by each method. Comparing the average of 95th percentile values, the traditional method's average was approximately 1.3 times higher than that of the proposed method, significantly exceeding the results of the proposed method. Furthermore, when comparing the standard deviations, the proposed method exhibited less variability in results due to sample variability compared to the traditional method. Therefore, in terms of stress side estimation, the proposed method allows for a more robust estimation, providing an estimation that eliminates conservatism and is less susceptible to sample variability than the traditional approach.

3.2.2. Estimation of the Strength Side

Using the median values for the parameters of the fracture probability curve, the traditional fracture limit, defined at 15% ECR, resulted in a significantly low fracture probability of 1.95×10^{-5}%. This finding indicates that the proposed method can calculate a more realistic fracture limit that eliminates conservatism compared to the traditional method.

3.2.3. Fracture Determination

Finally, fracture determination was performed using both the proposed and traditional methods based on the BEPU results for the PWR LBLOCA scenario. The traditional method applied order statistics of the 40th order to the stress side estimation, using the virtual dataset of 1020 samples as is.

The fracture probability was estimated to be approximately 15.1% using the proposed method. In contrast, the results of the traditional method showed a 95/95 upper tolerance limit of approximately 27.5% ECR, significantly exceeding the Japanese ECCS acceptance criterion (15% ECR) and thus determined as a fracture.

This comparison demonstrates that the proposed method, which calculates specific fracture probabilities, provides a more detailed outcome than the binary determination of the traditional method. Additionally, even in situations where the 95/95 upper tolerance limit considerably exceeds the ECCS acceptance criterion—traditionally determined as a fracture—the actual fracture probability is found to be about 15.1%. This indicates that the proposed method eliminates the conservatism built into both the deterministic fracture criterion and the stress side estimation using order statistics, allowing for a more accurate core damage determination aligned with actual conditions.

Table 1. Comparison of the 95th Percentiles Calculated by Each Method

	Average (ECR, %)	**Standard deviation (ECR, %)**	**Percentile (-)**
Proposed method	28.2	2.51	96.4
Traditional method	36.5	11.6	99.5

4. IMPROVING COMPUTATIONAL ACCURACY OF RARE EVENTS

4.1. Fracture Probability Calculation Using Numerical Integration

Our proposed method employed a Monte Carlo simulation to calculate the fracture probability through the stress-strength model; however, this method presents challenges in accurately estimating fracture probabilities of low-frequency events. Solutions to this challenge include importance sampling [9], which weights low-frequency events more heavily, or replacing the Monte Carlo simulation with numerical integration to calculate the fracture probability.

This chapter explores the calculation of the fracture probability using numerical integration. This approach involves integrating the overlapping areas of the stress and strength distributions. Due to the complexity of deriving an analytical solution, the ECR domain from 0 to 1 was divided into sufficiently small intervals ($1/10^5$), and numerical integration using the trapezoidal rule was applied to find an approximate solution.

This method could estimate fracture probabilities of low-frequency events with lower computational loads than the Monte Carlo simulation. However, using the numerical integration, it is not possible to directly handle distributions that include uncertainties. Therefore, the confidence levels of the parameters of the stress and strength distributions were set to specific values, and the uncertainties were removed prior to use. These results were then compared with those obtained from the Monte Carlo simulation.

The calculation of fracture probability using the numerical integration is expressed as follows:

$$P = \int_0^1 [f_{stress}(x|\mu,\sigma) \times F_{strength}(x|\alpha,\beta)]dx \tag{6}$$

where $f_{stress}(x|\mu,\sigma)$ represents the probability density function of ECR for the stress side, $F_{strength}(x|\alpha,\beta)$ represents the cumulative probability distribution function of ECR for the strength side, x represents ECR, and other parameters represent those estimated via Bayesian inference, set to a specific confidence level.

4.2. Monte Carlo Simulation vs. Numerical Integration

To compare accuracy of fracture probabilities relative to the number of trials of the Monte Carlo simulation, fracture probabilities were calculated using both the Monte Carlo simulation and the numerical integration. This comparison was made using known stress and strength distributions such that the true value of the fracture probability is 1.0×10^{-2}%. For the stress side ECR distribution, a log-normal distribution with a mean of -2.64 and a standard deviation of 0.3 was used. For the strength side ECR distribution, the distribution shown in Figure 2 with median values of the parameters was used. Furthermore, to evaluate the sample-derived variation of fracture probability for the Monte Carlo simulation, the fracture probability was calculated 100 times for each trial, estimating both the average and standard deviation of the fracture probability.

Table 2 shows the fracture probabilities calculated using Monte Carlo simulation and numerical integration. Additionally, to visualize variability for each number of trials, Figure 3 presents violin plots combined with box plots. From these results, it was confirmed that the Monte Carlo simulation, when the order of the number of trials was not sufficiently larger than the order of the reciprocal of the fracture probability (10^4 trials), resulted in significant variability in outcomes. In contrast, when using numerical integration, the fracture probability was accurately calculated. Therefore, it is concluded that the numerical integration, in comparison to the Monte Carlo simulation, can accurately estimate the fracture probabilities of low-frequency events with superior computational efficiency.

Table 2. Monte Carlo Simulation vs. Numerical Integration

	Fracture Probability (%)		
	10^3 trials	**10^4 trials**	**10^5 trials**
Monte Carlo Simulation (Average / SD, %)	4.0×10^{-3} / 2.1×10^{-2}	1.0×10^{-2} / 9.9×10^{-3}	1.0×10^{-2} / 3.0×10^{-3}
Numerical Integration (%)	1.0×10^{-2}	1.0×10^{-2}	1.0×10^{-2}

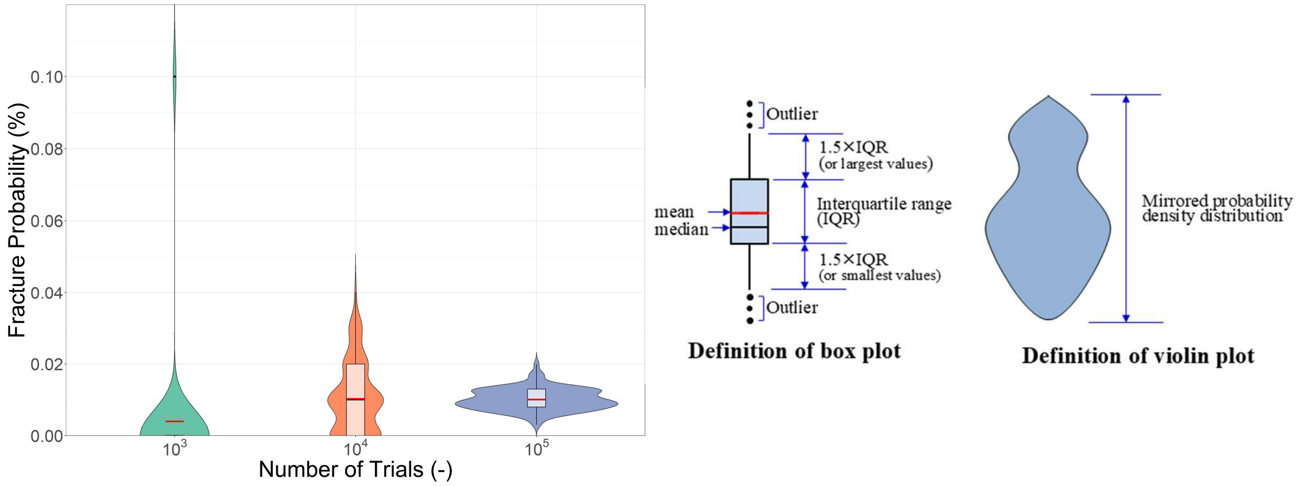

Figure 3. Variability of Fracture Probability Estimated by Monte Carlo Simulation

4.3. Consideration of Uncertainty in Numerical Integration

In Section 4.2, we demonstrated the numerical integration enabled the estimation of fracture probabilities for low-frequency events with lower computational loads compared to the Monte Carlo simulation. However, this result was obtained using the median values of parameters for both the stress and strength distributions. Therefore, the challenge in numerical integration lies in accounting for the uncertainties in both distributions. To address this challenge, we investigated the confidence levels of parameters for both stress and strength distributions to determine at which confidence levels the fracture probabilities calculated by numerical integration would match those obtained by Monte Carlo simulation that takes into account the overall uncertainties in both distributions.

Table 3 compares the fracture probability estimated using numerical integration with parameters set at various confidence levels and Monte Carlo simulation. The same virtual dataset used in Chapter 3 was used for this comparison.

Numerical integration with parameters set at a 55% confidence level estimated the fracture probability close to that obtained from Monte Carlo simulation, which takes into account the overall uncertainties in both stress and strength distributions. Thus, by setting appropriate confidence levels for the parameters of the stress and strength distributions, it is possible to calculate fracture probabilities that consider uncertainties using numerical integration. This approach allows for estimating fracture probabilities with high computational accuracy and low computational load while considering uncertainties, even for low-frequency events.

Table 3 also shows that the fracture probability estimated by numerical integration with parameters set at a 99% confidence level was 27.1%. Considering that the traditional deterministic method judges this virtual dataset as fractured as shown in Section 3.2.3, this result clearly demonstrates the inherent conservatism of the traditional method.

Table 3. Monte Carlo Simulation vs. Numerical Integration with Parameters at Various Confidence Levels

	Monte Carlo Simulation	Numerical Integration				
		50% confidence level	55% confidence level	70% confidence level	90% confidence level	99% confidence level
Fracture Probability (%)	15.1	14.6	15.1	17.0	21.0	27.1

5. CONCLUSION

This study aims to propose a probabilistic approach for determining core damage during LOCAs through the integration of the fuel fracture probability estimation model and the plant response analysis with the BEPU approach. Our proposed approach estimates the fracture probability using the stress-strength model and Monte Carlo simulations. To evaluate the effectiveness of our approach, we conducted numerical experiments and compared the results with the existing deterministic approach. Additionally, we explored numerical integration to enhance the accuracy of fracture probability estimation for low-frequency events, offering an alternative to Monte Carlo simulation, which may not effectively handle such events.

The numerical experiments demonstrated that our proposed method allows for a more rational and realistic determination of fuel rod fractures, eliminating the conservatism inherent in the traditional deterministic method. The comparison of fracture probabilities between numerical integration and Monte Carlo simulation showed that by setting parameters at appropriate confidence levels, numerical integration can accurately estimate fracture probabilities, considering the overall uncertainties in both stress and strength distributions, with a lower computational load.

Looking ahead, we consider employing importance sampling techniques that assign higher weights to low-frequency samples within the same distribution handled by Monte Carlo simulation, enabling the estimation of fracture probabilities for low-frequency events with fewer trials. While this study modeled fuel fracture, future efforts could model the entire process from fuel fracture to core damage. This would shift from the conservative evaluation of “fuel fracture equals core damage” to a more realistic evaluation. Potential applications could include using the integrity of the reactor’s pressure vessel to determine core damage.

This study evaluated the fracture probability of fuel rods based on ECR, whereas many conventional static PRA adopt a single numerical value of PCT or core water level as the core damage criterion. This is due to the need to reduce computational load and the belief that even if ECR is evaluated and the exact time of core damage is estimated, it would only bring about a small change in human error probability for accident management. However, the proposed method allows for a more realistic estimation of the time margin until core damage while keeping computational load low. Quantitatively assessing how this affects the success probability of accident management could further demonstrate the effectiveness of the proposed method.

Furthermore, the approach proposed in this study can be applied to core damage determination using PCT, enabling more rational core damage estimation. In dynamic PRA, which explicitly considers the time evolution of events, the proposed method, capable of more precise estimation of core damage timing, is expected to be even more effective.

Regarding the implementation of the proposed method, in addition to improving computational efficiency, it is necessary to consider efficient implementation methods, such as selective application to high-importance sequences. This approach would allow for enhanced accuracy in critical scenarios while managing overall computational demands in large-scale PRA.

References

[1] Nagase F et al. Technical Basis of ECCS Acceptance Criteria for Light-water Reactors and Applicability to High Burnup Fuel. JAEA-Review 2020-076, 2021.
[2] Narukawa T et al. Experimental and statistical study on fracture boundary of non-irradiated Zircaloy-4 cladding tube under LOCA conditions. Journal of Nuclear Materials, 499, 528-538, 2018.
[3] Zugazagoitia E et al. Uncertainty and sensitivity analysis of a PWR LOCA sequence using parametric and non-parametric methods. Reliability Engineering and System Safety, 193, 106607, 2020.
[4] NRC. TRACE V5.840, user’s manual: input specification, 2014.
[5] Matsuura K. “Wonderful R 2, Bayesian Statistical Modeling Using Stan and R”, Kyoritsu Shuppan Co., Ltd, Tokyo, Japan, 2016 [in Japanese].
[6] Baker L and Just L C. Studies of metal-water reactions at high temperatures. III. Experimental and theoretical studies of the zirconium-water reaction. Argonne National Laboratory, ANL-6548, 1961.
[7] Cathcart J V et al. Zirconium metal-water oxidation kinetics. IV. Reaction rate studies. Oak Ridge National Laboratory, ORNL/NUREG-17, 1977.

[8] Wilks S S. The large-sample distribution of the likelihood ratio for testing composite hypotheses. The Annals of Mathematical Statistics, 9, 1, 60-62, 1938.

[9] Rubin D B. The calculation of posterior distributions by data augmentation: comment: a noniterative sampling/importance resampling alternative to the data augmentation algorithm for creating a few imputations when fractions of missing information are modest: The SIR algorithm. Journal of the American Statistical Association, 82, 398, 543-546, 1987.