Distribution-free stochastic model updating for the Physics-guided reliability analysis of a material thermal property under limited data

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Abstract: The paper presents a distribution-free stochastic model updating approach to address a reliability problem based on the 2008 Sandia thermal problem involving a material used for the safety-critical components in a nuclear reactor. One of the challenges involves quantifying the thermal reliability of the material, along with its uncertainties, given only limited experimental data. To achieve this, the proposed methodology involves the moment-matching staircase density function to characterise the variability of the aleatory input model parameters based on the available data. To calibrate the staircase density function on the respective aleatory parameters, the approximate Bayesian computation technique is implemented, along with the Jensen-Shannon divergence as the distance function and the Transitional Ensemble Markov Chain Monte Carlo to provide posterior estimates on the parameters of the staircase density function. In doing so, it removes the assumption on the distribution class associated with the aleatory characteristics of the input model parameters. This yields a probability box on the aleatory parameters which is then propagated through the physics-guided performance function of the material which yields an imprecise probability on the thermal reliability of the material. To demonstrate the feasibility and verify the proposed method, the results are compared against published results to the problem.

Keywords: Bayesian model updating, Transitional Ensemble Markov Chain Monte Carlo, Staircase density function, Imprecise probability.

1. INTRODUCTION

An aspect of nuclear safety is the reliability of the materials used to construct the safety-critical components of a nuclear reactor. As such, it is important that a reliability analysis is performed on such materials to ensure that the risk of a nuclear-related accident due to a structural compromise is minimised. Such study constitutes part of the Level 1 Probabilistic Safety Assessment of the nuclear reactor.

More often, it is inevitable that uncertainties should be considered and characterized when performing reliability analysis. Such uncertainties are generally categorized into two types: 1) aleatory uncertainty; and 2) epistemic uncertainty [1]. In the paper, the problem of polymorphic uncertainty is investigated where both types of uncertainty are present simultaneously. Polymorphic uncertainties are characterised using imprecise probability models, an example to which is the probability box which will be implemented in the paper [2]. Often, the choice of distribution model is made under physical or empirical assumptions. However, such choices may not truly reflect the true variability of the given parameter(s). As such, a distribution-free approach involving the Staircase Density Function (SDF) is used to remove model uncertainty over the distribution class and provide a general approach towards characterising the polymorphic uncertainty in reliability analysis under limited data [3].

The proposed approach involves performing stochastic model updating on the shape parameters of the SDF via a distance-based approximate Bayesian computation incorporating the Jensen-Shannon divergence. The research aim is to demonstrate the feasibility of the proposed framework by implementing it towards addressing the reliability problem presented in the 2008 Sandia thermal problem under limited data [4]. To achieve the research objective, the paper first introduces the Bayesian model updating framework and the distance-based approximate Bayesian computation using the Jensen-Shannon divergence. Following which, the mathematical concept of the SDF is presented, and this is proceeded with the presentation of the Sandia Thermal problem. From there, the results are obtained and discussed, thereby evaluating the proposed method. Finally, the paper concluded with a summary of the contents presented and recommendations for future work.

2. BAYESIAN MODEL UPDATING

A widely implemented approach towards stochastic model updating is Bayesian model updating which is mathematically defined as [1]:

$$P(\boldsymbol{\theta}|\boldsymbol{D},M) = \frac{P(\boldsymbol{\theta}|M)\cdot P(\boldsymbol{D}|\boldsymbol{\theta},M)}{P(\boldsymbol{D}|M)} \tag{1}$$

where $P(\boldsymbol{\theta}|M)$ is the prior distribution reflecting the prior knowledge on the inferred parameter(s) $\boldsymbol{\theta}$ before collecting data $\boldsymbol{D}$, $P(\boldsymbol{D}|\boldsymbol{\theta},M)$ is the likelihood function reflecting the degree of agreement between the observed data $\boldsymbol{D}$ and the prediction from model M given $\boldsymbol{\theta}$, and $P(\boldsymbol{D}|M)$ is the evidence which ensures that the posterior integrates to one. Details on each of the above terms in Eq. (1) can be found in Lye et al. (2023) [1]. Generally speaking, the inferred parameter(s) may be time-invariant, or time-varying [5]. For the paper, the inferred parameters on interest are time-invariant.

However, given that $P(\boldsymbol{D}|M)$ is a numerical constant, the term is usually neglected thereby re-expressing the posterior in its un-normalised form:

$$P(\boldsymbol{\theta}|\boldsymbol{D},M) \propto P(\boldsymbol{\theta}|M)\cdot P(\boldsymbol{D}|\boldsymbol{\theta},M) \tag{2}$$

As such, the direct Monte Carlo sampling technique becomes inapplicable thereby bringing forth the need for advanced sampling techniques [6]. For the work presented in the paper, the state-of-the-art Transitional Ensemble Markov Chain Monte Carlo (TEMCMC) method will be implemented owing to its effectiveness in sampling from highly skewed, anisotropic posterior distributions [7].

2.1. Transitional Ensemble Markov Chain Monte Carlo

The TEMCMC sampler is a variant of the Transitional Markov Chain Monte Carlo (TMCMC) sampler originally developed by Ching and Chen (2007) [8]. A key characteristic of the TMCMC sampling approach is that it generates samples from complex-shaped posteriors (e.g., very peaked, or multi-modal) in an iterative manner. This is achieved through a series of intermediate functions known as transitional distributions P^j which is defined as:

$$P^j \propto P(\boldsymbol{\theta}|M)\cdot P(\boldsymbol{D}|\boldsymbol{\theta},M)^{\beta_j} \tag{3}$$

where $j \geq 0$ is the sampling iteration number, β_j is the tempering parameter such that $0 = \beta_0 < \beta_1 < \ldots < \beta_{m-1} < \beta_m = 1$, and m is the final iteration number. This allows for P^j to transit gradually from the prior to the posterior.

The sampling procedure follows: At iteration $j = 0$, samples are generated from the prior via direct Monte Carlo sampling. At iteration $j = 1$, the algorithm computes β_j, and samples from iteration $j - 1$ are updated according to P^j via the Affine-invariant Ensemble Markov Chain Monte Carlo sampler. From there, the algorithm proceeds to iteration $j = j + 1$ where the sample updating procedure repeats until the last iteration $j = m$. Full details on the TEMCMC sampler and its algorithm are found in [7].

2.2. Approximate Bayesian Computation

An essential component of the Bayesian model updating procedure is the definition of the likelihood function $P(\boldsymbol{D}|\boldsymbol{\theta},M)$ as seen in Eq. (1). Assuming independence between the N_{obs} observations, the full analytical likelihood is defined as follows:

$$P(\boldsymbol{D}|\boldsymbol{\theta},M) = \prod_{k=1}^{N_{\text{obs}}} P(\boldsymbol{D}_k|\boldsymbol{\theta},M) \tag{4}$$

However, the evaluation of the full analytical likelihood function in Eq. (4) can be computationally demanding since it requires a large amount of model evaluations. Such issue becomes significantly pronounced when the model M is computationally expensive. To address such issue, the Approximate

Bayesian computation approach is implemented by substituting the full likelihood function in Eq. (4) with the distance based approximate likelihood function proposed by Bi et al. (2018) [9]:

$$P(\boldsymbol{D}|\boldsymbol{\theta}, M) = \exp\left[-\frac{d^2}{\varepsilon^2}\right] \tag{5}$$

where d is the distance function serving to quantify the statistical difference between the distribution of the data $\boldsymbol{D}$ and that of the output from model M, while ε is the width-factor which controls the centralization degree of the resulting posterior. As a guide, the width factor should lie within the interval of $[10^{-3}, 10^{-1}]$. For the work presented in the paper, the Jensen-Shannon divergence is implemented as the choice of the distance function [10].

2.3. Jensen-Shannon divergence

The Jensen-Shannon divergence serves to quantify the difference in distribution between two statistical objects using information entropy and is based on the Kullback-Leibler divergence in which the latter is defined as [11]:

$$d_{\mathrm{KL}}(p_1 \,||\, p_2) = \sum_{x_d=1}^{N_{\mathrm{bin}}} \cdots \sum_{x_1=1}^{N_{\mathrm{bin}}} p_1\left(b_{x_1,\ldots,x_d}\right) \cdot \log\left[\frac{p_1\left(b_{x_1,\ldots,x_d}\right)}{p_2\left(b_{x_1,\ldots,x_d}\right)}\right] \tag{6}$$

where N_{bin} is the total number of bins used to approximate the distributions p_1 and p_2. However, the implementation Kullback-Leibler divergence as the distance function would not be optimal for ABC for the following reasons: 1) it does not obey the symmetrical property (i.e., $d_{\mathrm{KL}}(p_1 \,||\, p_2) \neq d_{\mathrm{KL}}(p_2 \,||\, p_1)$); and 2) the function yields infinity when the support of p_1 is not a subset of p_2. Hence, the Jensen-Shannon divergence was developed to overcome such drawbacks and is defined as:

$$d_{\mathrm{JS}}(p_1, p_2) = \frac{1}{2} \cdot \left(d_{\mathrm{KL}}(p_1 \,||\, T) + d_{\mathrm{KL}}(p_2 \,||\, T)\right), \qquad \text{for } \mathrm{T} = \frac{1}{2} \cdot (p_1 + p_2) \tag{7}$$

In the context of ABC, the interest would be to compute $d_{\mathrm{JS}}(p_M, p_{\boldsymbol{D}})$ where p_M is the distribution of the model prediction while $p_{\boldsymbol{D}}$ is the distribution of the observed data. Its implementation for Approximate Bayesian computation was proposed by Yang et al. (2022) and further studied by Lye et al. (2024) to perform model calibration and validation for a black-box system under hybrid uncertainties and limited data [10, 11].

An essential component of the Jensen-Shannon divergence is the parameter N_{bin}. To determine empirically the optimal number of bins, the adaptive-binning algorithm proposed by Zhao et al. (2022) is implemented to which the procedure follows [12]:

1) Compute the parameter Δ^{sim} following:

$$\Delta^{\mathrm{sim}} = \max\left(\max\left|\boldsymbol{D}_{i,m}^{\mathrm{sim}} - \boldsymbol{D}_{j,m}^{\mathrm{sim}}\right|\right) \tag{8}$$

   where $i,j = 1,\ldots,N$ and $m = 1,\ldots,d$. Note that N is the total number of model evaluations corresponding to the total sample size from the posterior while $\boldsymbol{D}^{\mathrm{sim}}$ is the simulated data (i.e., model prediction);
2) Compute the Euclidean distance d_E between $\boldsymbol{D}$ and $\boldsymbol{D}^{\mathrm{sim}}$ following:

$$d_E = \sqrt{\left(\overline{\boldsymbol{D}^{\mathrm{sim}}} - \overline{\boldsymbol{D}}\right) \cdot \left(\overline{\boldsymbol{D}^{\mathrm{sim}}} - \overline{\boldsymbol{D}}\right)^{\mathrm{T}}} \tag{9}$$

   where $\overline{\boldsymbol{D}^{\mathrm{sim}}}$ and $\overline{\boldsymbol{D}}$ are the means of the simulated data and that of the observed data respectively;
3) Compute the bin width parameter w following:

$$w = \frac{\log\left[\Delta^{\mathrm{sim}} + 1\right]}{\max\left(N^{\frac{1}{3}}, N_{\mathrm{obs}}^{\frac{1}{3}}\right)} \times \exp[d_E] \tag{10}$$

4) Finally, compute the number of bins N_{bin} following:

$$N_{\text{bin}} = \left(\left\lceil \frac{\Delta^{\text{sim}}}{w} \right\rceil\right)^{d} \tag{11}$$

As an adaptive variable, the number of bins is bounded such that:

$$2^{d} \leq N_{\text{bin}} \leq \left(\left\lceil \frac{\max(N,\ N_{\text{obs}})}{10} \right\rceil\right)^{d} \tag{12}$$

3. STAIRCASE DENSITY FUNCTION

For the work presented in the paper, the hybrid uncertainties over the physical parameters of interest are characterized using the SDF. The SDF is a moment-matching meta-model developed by Crespo et al (2018) that models a given data distribution based on its r^{th} central moment m_r defined as [13]:

$$m_r = \int_{\underline{z}}^{\overline{z}} (z-\mu)^r \cdot f_z(z) \cdot dz \ , r = 0,1,2,\ldots \tag{13}$$

where the integration limits $\Delta_z = [\underline{z}, \overline{z}]$ constitutes the bounded support set over the staircase random variable z, the function f_z is the density function, and μ is the expected value of the data variable z. It needs to be highlighted that $m_0 = 1$, $m_1 = 0$, m_2 is the variance, m_3 is the third-order central moment, and m_4 is the fourth-order central moment. The parameters constituting the staircase random variable are: $\boldsymbol{\theta}_z = \{\Delta_z, \mu, m_2, m_3, m_4\}$ which are constrained following a series of inequalities: $\boldsymbol{\Theta} = \{\boldsymbol{\theta}_z: g(\boldsymbol{\theta}_z) \leq 0\}$ derived from 14 moment constraints on which details are found in Crespo et al. (2018) [13]. Based on the constraints, for a given support set Δz, the feasible intervals of μ, m_2, m_3, and m_4 are as follows:

$$\mu \in [\underline{z}, \overline{z}], \quad m_2 \in \left[0, \frac{(\overline{z}-\underline{z})^2}{4}\right], \quad m_3 \in \left[-\frac{(\overline{z}-\underline{z})^3}{6\sqrt{3}}, \frac{(\overline{z}-\underline{z})^3}{6\sqrt{3}}\right], \quad m_4 \in \left[0, \frac{(\overline{z}-\underline{z})^4}{12}\right] \tag{14}$$

The density function f_z is defined mathematically as:

$$f_z = \begin{cases} h_{i_b} \ \forall\ z \in ((i_b-1)\cdot\kappa, i_b \cdot \kappa] \ , \text{for } 1 \leq i_b \leq N_b \\ 0 \qquad\qquad\qquad\qquad\quad , \text{otherwise} \end{cases} \tag{15}$$

where $N_b = 50$ denotes the number of bins, h_{ib} is the height of the SDF in the i_b^{th} bin, and $\kappa = (\overline{z}-\underline{z})/2$ is the length of each sub-interval. It is to be noted that $h_{ib} \geq 0$ for all N_b bins and that their values are obtained by solving the following convex optimization problem:

$$\widehat{h_{i_b}} = argmin_{\boldsymbol{h} \geq 0} \left\{ J(\boldsymbol{h}): \sum_{i_b=1}^{N_b} \int_{(i_b-1)\cdot\kappa}^{i_b\cdot\kappa} z \cdot h_{i_b} \cdot dz = \mu \ , \sum_{i_b=1}^{N_b} \int_{(i_b-1)\cdot\kappa}^{i_b\cdot\kappa} (z-\mu)^r \cdot h_{i_b} \cdot dz = m_r \ , r = 0, 2, 3, 4 \right\} \tag{16}$$

where $J(\boldsymbol{h})$ is the cost-function defined as:

$$J(\boldsymbol{h}) = \boldsymbol{h}^T \boldsymbol{I} \boldsymbol{h} \tag{17}$$

for which $\boldsymbol{I}$ is the identity matrix. The cost-function defined in Eq. (17) would yield a staircase random variable which minimises the squared sum of the likelihood at each bin [3].

4. CASE STUDY: SANDIA THERMAL PROBLEM

The case study is based on the 2008 Validation Challenge workshop hosted by Sandia National Laboratories to which full details can be found in Dowding et al. (2008) [4]. The objectives of the case study are: 1) to characterise the variability of the aleatory variables using the SDF and evaluate its effectiveness; and 2) to analyse the reliability of a given material thermal property against a regulatory requirement.

4.1. Background

The set-up involves a slab material which can be used to construct a nuclear reactor vessel containment structure. The thermal property of the slab material is such that its temperature response T under different heating conditions is mathematically defined from physics as a truncated infinite series following [4]:

$$T(x,t) = \begin{cases} T_i & ,\text{for } t = 0s \\ T_i + \frac{qL}{k}\left[\frac{\left(\frac{k}{\rho C_p}\right)t}{L^2} + \frac{1}{3} - \frac{x}{L} + \frac{1}{2}\left(\frac{x}{L}\right)^2 - \frac{2}{\pi^2}\sum_{n=1}^{6}\frac{1}{n^2}\exp\left(-n^2\pi^2\frac{\left(\frac{k}{\rho C_p}\right)t}{L^2}\right)\cos\left(n\pi\frac{x}{L}\right)\right] & ,\text{for t} > 0s \end{cases} \quad (18)$$

where $T_i = 25^oC$ is the initial ambient temperature, $L = 0.0190\ m$ is the thickness of the slab, x is the location variable along the thickness of the slab, t is the time since the start of the heating process, q is the heat flux, k is the thermal conductivity of the slab, and ρC_p is the heat capacity of the material. The boundary conditions are constant heat flux on the $x = 0\ m$ face and adiabatic on the $x = L$ face.

The regulatory requirement on the material stipulates that at time $t' = 1000\ s$ after the heat flux exposure of $q = 3500\ W/m^2$, the probability that the surface temperature of the slab T_s (i.e., at $x = 0\ m$) exceeds $T_f = 900\ ^oC$ must be less than the threshold probability value $p_f = 0.01$ such that:

$$P\left(T_s(t = t') > T_f\right) < p_f \quad (19)$$

The aleatory variables are the material thermal properties k and ρC_p and it is assumed that the two parameters are independent from one another. Hence, the need to characterise the variability of the two parameters which is done based on a set of 20 experimental data, for each parameter, obtained across five different temperature response values: $T = \{20, 250, 500, 750, 1000\}\ ^oC$. The corresponding numerical data for each variable is presented in Table 1 while their corresponding scatter plot diagrams are illustrated in Figure 1.

Table 1. Numerical data for each material property given each value of T.

$T\ [^oC]$	20	250	500	750	1000
k	0.0496	0.0628	0.0602	0.0657	0.0631
	0.0530	0.0620	0.0546	0.0713	0.0796
$[W/m\ ^oC]$	0.0493	0.0537	0.0628	0.0694	0.0692
	0.0455	0.0561	0.0614	0.0732	0.0739
ρC_p	3.76×10^5	3.87×10^5	4.52×10^5	4.68×10^5	4.19×10^5
	3.38×10^5	4.69×10^5	4.10×10^5	4.24×10^5	4.38×10^5
$[J/m^3\ ^oC]$	3.50×10^5	4.19×10^5	4.02×10^5	3.72×10^5	3.45×10^5
	4.13×10^5	4.28×10^5	3.94×10^5	3.46×10^5	3.95×10^5

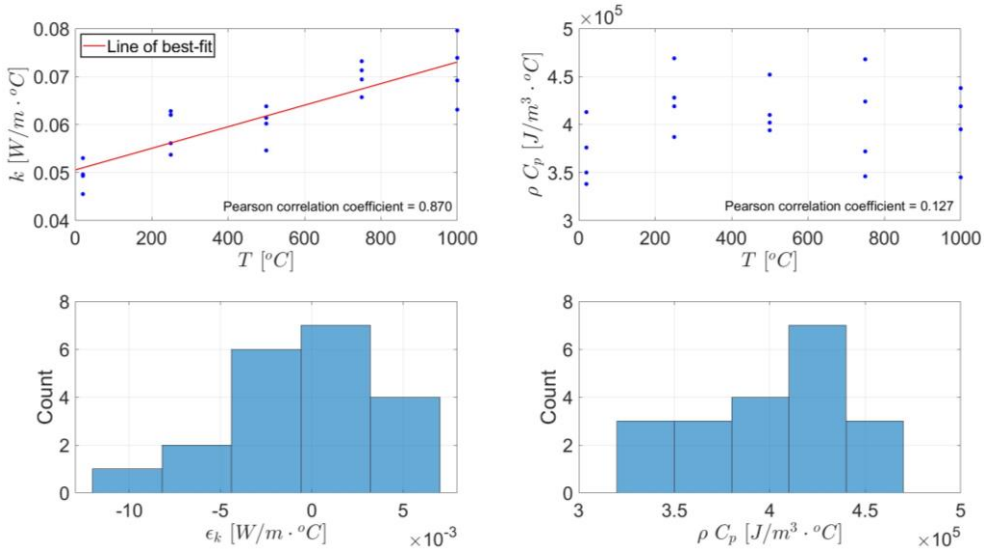

Figure 1. Scatter plot diagrams for k and ρC_p, along with histograms for ϵ_k and ρC_p.

The problem is divided into two parts: 1) to characterise the polymorphic uncertainty associated with the material thermal properties k and ρCp given the limited data set; and 2) to perform the reliability analysis on the thermal properties of the given slab material.

4.2. Polymorphic uncertainty characterisation

Based on the scatter plots presented in Figure 1, there is a significantly strong correlation between k and T where the Pearson correlation coefficient between the two quantities is at 0.870. Such observation is consistent with the underlying physics and presents the need to account for the relationship between k and T. To do so, a linear regression is done to model such relationship and the resulting linear model is defined as [14]:

$$k(T) = 2.249 \times 10^{-5} \cdot T + 0.051 + \epsilon_k \tag{20}$$

where ϵ_k is the residual of the linear model for which its data is obtained from Eq. (20) using the data for k and T in Table 1. It is assumed that ϵ_k is independent of temperature.

From the scatter plot in Figure 1, it is observed that that is a relatively weak correlation between ρC_p and T where the Pearson correlation coefficient between the two quantities is at 0.127. As such, the dependency between the two quantities can be neglected and they are taken to be independent from one another. The histogram representation of the distribution of ϵ_k and ρC_p are illustrated in Figure 1.

Given no information over the distribution class of the aleatory variables ϵ_k and ρC_p, the SDF is used to model the distribution of the two parameters. For each quantity, the Bayesian model updating framework is implemented to infer the shape parameters of the SDF, namely: $\boldsymbol{\theta} = \{\mu, m_2, (m_3/(m_2)^{3/2}), (m_4/(m_2)^2)\}$; where $(m_3/(m_2)^{3/2})$ is the skewness parameter while $(m_4/(m_2)^2)$ is the kurtosis parameter. The support set of the SDF for ϵ_k is set at $\Delta\epsilon_k$ = [-0.02, 0.02] $W/m\ ^oC$ while that for ρC_p is set at $\Delta_{\rho Cp}$ = [300000, 500000] $J/m^3\ ^oC$. For each of the aleatory variable, the prior distributions on the SDF parameters $\boldsymbol{\theta_z}$ are set as Uniform distributions whose respective bounds are defined as per Eq. (14) and presented in Table 2.

Table 2. Uniform prior bounds to the SDF parameters for ϵ_k and ρC_p.

Parameters	ϵ_k	ρC_p
μ	[-0.02, 0.02] $W/m\ ^oC$	[300000, 500000] $J/m^3\ ^oC$
m_2	[0, 1.60×10^{-3}] $(W/m\ ^oC)^2$	[0, 1.00×10^{10}] $(J/m^3\ ^oC)^2$
m_3	[- $(3.20 \times 10^{-5})/3\sqrt{3}$, $(3.20 \times 10^{-5})/3\sqrt{3}$] $(W/m\ ^oC)^3$	[- $(4.00 \times 10^{15})/3\sqrt{3}$, $(4.00 \times 10^{15})/3\sqrt{3}$] $(J/m^3\ ^oC)^3$
m_4	[0, $(6.4 \times 10^{-7})/3$] $(W/m\ ^oC)^4$	[0, $(4.00 \times 10^{20})/3$] $(J/m^3\ ^oC)^4$

The likelihood function is defined as per Eq. (5). For the case of inferring $\boldsymbol{\theta_z}$ to model the SDF for ϵ_k, the width parameter is set at ε = 0.050 $W/m\ ^oC$ whereas for the case of inferring $\boldsymbol{\theta_z}$ to model the SDF for ρC_p, the width parameter is set at ε = 0.008 $J/m^3\ ^oC$. The rationale behind the choice of such width parameter values is to provide sufficient convergence on the posterior estimates of the inferred parameters – i.e., 5 sampling iterations for the case of ϵ_k and 6 iterations for the case of ρC_p by the TEMCMC sampler.

4.3. Results and discussions

The resulting histogram representation of the posterior sample distribution of the respective inferred parameter is presented in Figure 2. From which, a risk-based estimate on the reduced epistemic bounds of the inferred parameters is obtained by considering the credible intervals at alpha-level of 0.86 for the shape parameters to the SDF for ϵ_k and that for ρC_p. The resulting credible interval estimates obtained for the respective inferred parameter are presented in Table 3.

Table 3. Results to the credible interval estimates to the SDF parameters for ϵ_k and ρC_p.

Parameters	ϵ_k	ρC_p
μ	[-1.24×10^{-3}, -1.06×10^{-3}] $W/m\ ^oC$	[4.00×10^5, 4.01×10^5] $J/m^3\ ^oC$
m_2	[7.44×10^{-5}, 7.72×10^{-5}] $(W/m\ ^oC)^2$	[2.99×10^9, 3.27×10^9] $(J/m^3\ ^oC)^2$
$m_3/(m_2)^{3/2}$	[0.03, 0.19]	[-0.21, -0.07]
$m_4/(m_2)^2$	[4.03, 4.22]	[2.37, 2.52]

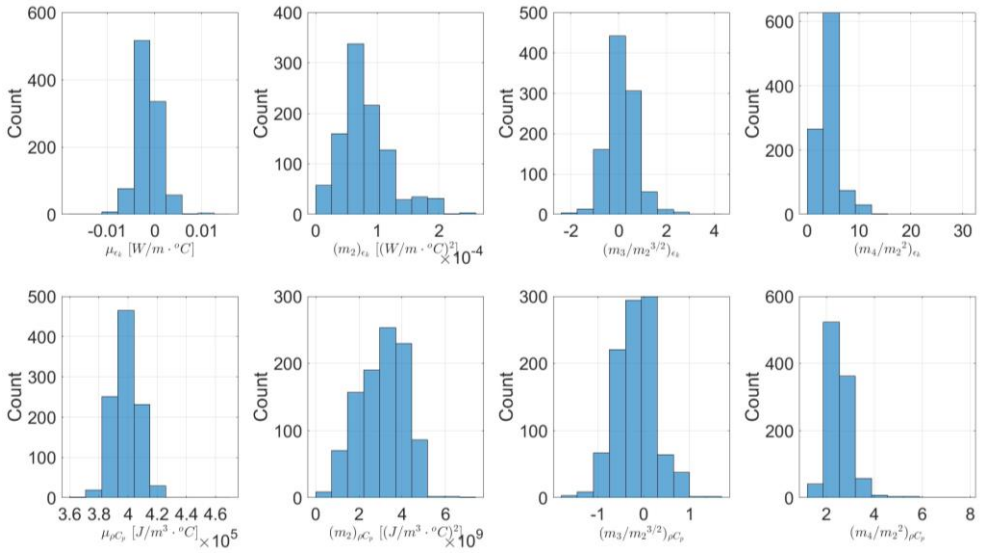

Figure 2. Histogram to the posterior samples obtained for the corresponding inferred parameters of the SDF.

From the above results to the credible interval estimates to the SDF shape parameters, a probability box is constructed over ϵ_k and ρC_p using Double-loop Monte Carlo. For each aleatory variable, a total of N_e epistemic samples are obtained uniformly from the epistemic four-dimensional hyper-rectangle whose bounds are defined by the resulting credible interval defined in Table 3. For each epistemic sample set realisation, a total of N_a aleatory samples are obtained from the resulting SDF using the given epistemic sample set as the input shape parameters. Repeating the generation of aleatory samples using all N_e epistemic sample set inputs, this produces N_e distinct SDF sample empirical cumulative distributions. The bounds to the empirical cumulative distribution ensemble constitute the probability box. The procedure is implemented for $\{N_e, N_a\} = \{1000, 10000\}$ to which the resulting probability box for ϵ_k and ρC_p are presented in Figure 3.

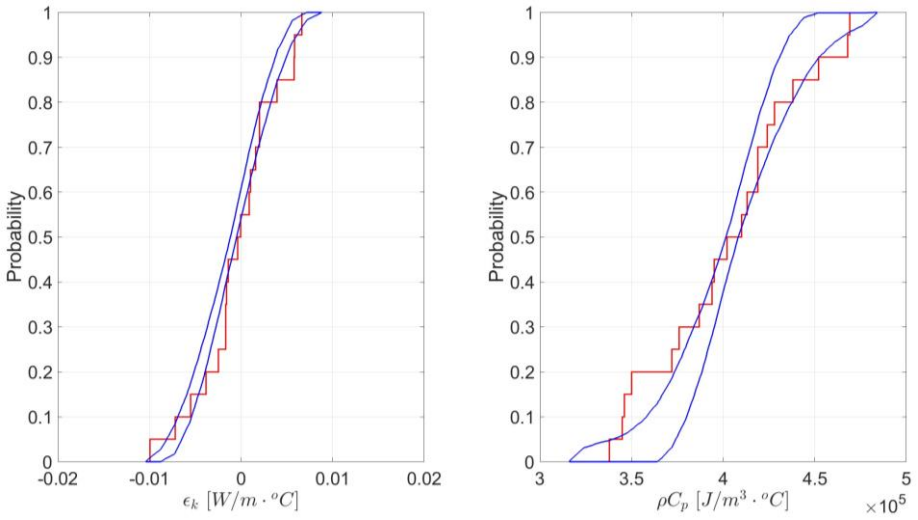

Figure 3. Probability box of ϵ_k and ρC_p (in blue) with the empirical cumulative distribution of the data (red).

Based on Figure 3, it is observed that the resulting probability boxes generally enclose the empirical cumulative distribution of the data for both ϵ_k and ρC_p which indicates that the distribution-free stochastic model updating approach is well-verified against the data provided.

4.4. Reliability analysis

For the reliability analysis, a performance function g is defined such that:

$$g(k, \rho C_p) = T(k, \rho C_p) - T_f \tag{21}$$

where $T(k, \rho C_p)$ is defined following Eq. (18). As seen in Eq. (21), the safe domain is defined as $g < 0$ while the failure domain is defined as $g > 0$. As such, the interest is to compute the failure probability $P(g > 0)$ whilst accounting for the polymorphic uncertainty due to k and ρC_p.

As seen in Eq. (18), the temperature T of the slab material is a function of k which itself is a function of the temperature T as seen in Eq. (20). This presents a system of two equations which can be solved iteratively based on the approach by Ferson et al. (2008) to compute the probability box of g as follows [14]: Firstly, a normal distribution is fitted over the entire 20 data points of k where the mean and standard deviation of the 20 data points serve as the shape parameter of the normal distribution. This is the unconditional distribution of k that is independent of T. From there, a Double-loop Monte Carlo procedure is implemented where for each N_e realization of the sample distribution of ϵ_k and ρC_p, the unconditional distribution of k is used to generate N_a seed samples of k as input from which an output distribution on T, given the current sample distribution of ρC_p, is obtained. This output distribution on T is then used as input to compute k via Eq. (20) given the current sample distribution of ϵ_k. The resulting distribution of k is then used to reseed the process until the distribution of T converges such that the area d_A enclosed by the empirical cumulative distribution of the current samples of T and that of the previous samples of T is sufficiently small. The convergence criteria is when $d_A < 0.6$, for which d_A is computed following [14]:

$$d_A = \int_{-\infty}^{\infty} |F_1(x) - F_2(x)| \cdot dx \tag{22}$$

where F_1 and F_2 are the previous and current empirical cumulative distribution functions of T respectively within which the area of interest is enclosed. Upon achieving convergence, the N_a values of g are computed based on the resulting sample distributions of k and ρC_p. The above procedure is repeated N_e times using all N_e sample distributions of k and ρC_p obtained from the analysis in Section 4.3. This generates N_e sample distributions of g, each of sample size N_a, from which a probability box of g is constructed following the procedure in Section 4.3 and is illustrated in Figure 4.

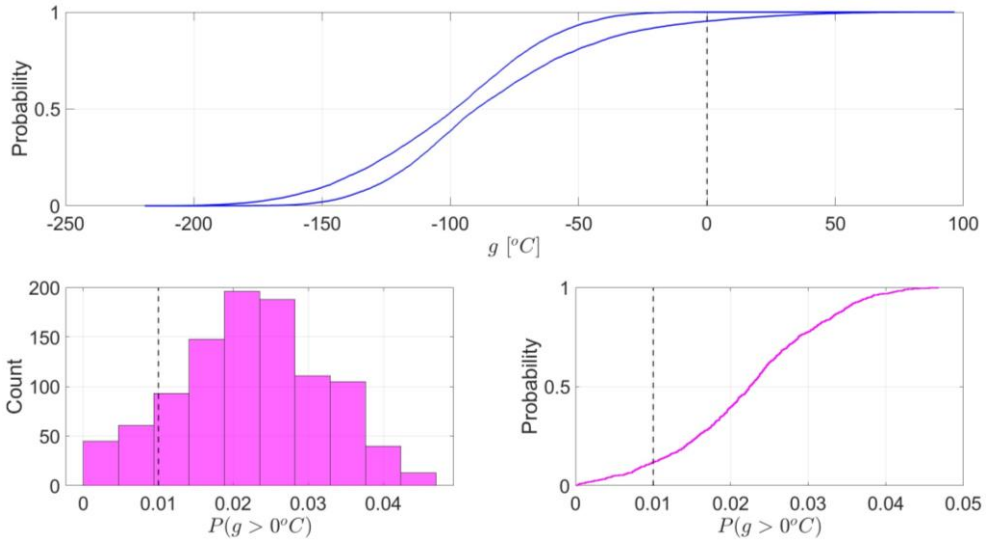

Figure 4. Empirical cumulative distribution of the performance function g along with the histogram and empirical cumulative distribution of the probability of not meeting the regulatory requirement across N_e runs.

4.5. Results and discussions

Based on the probability box illustrated in Figure 4, the resulting interval of probability $P(g > 0)$ is presented in Table 4 along with other published results as a form of comparison:

Table 4. Numerical results to the reliability analysis along with the corresponding reference.

Reference	Paper	**[14]**	**[15]**	**[16]**	**[17]**
$P(g > 0)$	[0, 0.05]	0.05	0.08	0.03	0.02

It can be seen from Table 4 that the probability interval of $P(g > 0)$ generally encloses the published results, with the exception of the result published by Brandyberry (2008) [15] whose result falls outside the resulting bounds. Such observation verifies and demonstrates the applicability of the proposed distribution-free stochastic model updating approach in quantifying the reliability of the given slab material surface temperature exceeding a threshold value under polymorphic uncertainty and limited data. However, it needs to be highlighted that the upper-bound value of $P(g > 0)$ is still above the regulatory requirement defined in Eq. (19) by four times.

An additional analysis is performed to obtain the probability of meeting the regulatory requirement. The following procedure is undertaken: For each of the N_e sample distributions of g, the probability $P(g > 0)$ is obtained. This allows for N_e values of $P(g > 0)$ to be obtained from which a histogram and the empirical cumulative distribution are illustrated in Figure 4. From the resulting empirical cumulative distribution of $P(g > 0)$, the probability of the regulatory requirement being met across N_e realisations of sample distribution of g is 0.120.

A possible reason for the the upper-bound value of $P(g > 0)$ exceeding the predefined requirement by a significant margin could be attributed to four factors: 1) the temperature model defined in Eq. (18); 2) the model that relates the temperature dependence of k; 3) the temperature-independent assumption on ρC_p; and 4) the independence assumption between k and ρC_p. For the first factor, the temperature model assumes that the input parameters are temperature-independent especially k and ρC_p. Such assumption is for the convenience of the analyst to reduce the complexity of the problem. It was found in Ferson et al. (2008) that the consideration of temperature dependence on the input parameter such as k would have a significant effect on the reduction on the probability $P(g > 0)$ [14]. For the second factor, a linear model was used to model the temperature dependence of k which may not be the true physics-based model. A linear model was chosen out of convenience based on the scatter plot profile between k and T provided in Figure 1. The choice of such temperature dependence model can have an effect on the computation of $P(g > 0)$ and this applies when considering the third factor. Finally, for the fourth factor, the independence assumption between k and ρC_p was made to reduce the complexity of the problem. However, it is to be acknowledged that there could be some form of dependency between the two parameters in reality which, if accounted for, would have a significant impact on the results to $P(g > 0)$.

5. CONCLUSION

The paper has proposed a distribution-free stochastic model updating framework to perform a physics-guided reliability analysis. It comprises of two key features: 1) the use of the Staircase Density Function to characterise the polymorphic uncertainty of model parameters without assuming any class of distribution model thereby removing the element of model uncertainty; and 2) the novel implementation of the Jensen-Shannon divergence as the distance function for the distance-based approximate Bayesian computation to perform the model updating procedure. To demonstrate the feasibility and robustness of the proposed approach, the 2008 Sandia thermal problem is used as the case study and application example which provides a realistic setting to a significant degree. The results to the reliability analysis show that the imprecise probability of the slab material exceeding a threshold temperature generally encloses the published values presented in Table 4. This verifies and validates the proposed distribution-free stochastic model updating approach.

Further research efforts can be invested towards the following: 1) comparing the reliability results using the proposed framework, but with different distance functions for the distance-based approximate Bayesian computation such as the Bhattacharyya and the Bray-Curtis distance functions; 2) to consider the physics-based temperature models for k and ρC_p and compare the results of the reliability analysis; and 3) to consider the uncertain dependency between k and ρC_p and propagate such uncertainty within the reliability analysis using the proposed approach.

To provide a better understanding to the proposed framework and to allow for the reproducibility of the results presented in the paper, the MATLAB codes to the study are made accessible on GitHub via: https://github.com/Adolphus8/stochastic-model-updating.git

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